Lipschitz Approximation to Corkscrew Domains and Harmonic Measure

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Theorem: (F. and M. Riesz 1916) Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, bounded by a Jordan curve.

If $\mathcal{H}^1(\partial \Omega) < \infty$, then the harmonic measure $\omega$ of $\Omega$ and the length measure $\sigma = \mathcal{H}^1 \llcorner \partial \Omega$ on the boundary are mutually absolutely continuous:

$$\omega(E) = 0 \iff \sigma(E) = 0 \iff \omega(E) = 0 \quad \text{for all } E \subset \partial \Omega$$

- In 1936 Lavrentiev showed that if $\partial \Omega$ is a chord arc curve, then $\omega \in A_{\infty}(\sigma)$.
- In 1990 Bishop and Jones proved “local” of F. and M. Riesz.

Question: What version of the F. and M. Riesz theorem holds on higher dimensional domains?
Harmonic Measure and Length \((n = 2)\)

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Harmonic Measure and Surface Measure \((n \geq 3)\)

Assume \(\Omega \subset \mathbb{R}^n\) and \(H^{n-1}(\partial \Omega) < \infty\). When is \(\omega \ll \sigma \ll \omega\)?

**Example: (Ziemer 1974)** There exists a topological ball \(\Omega \subset \mathbb{R}^3\) such that \(H^2(\partial \Omega) < \infty\) and a set \(E \subset \partial \Omega\) such that \(\sigma(E) > 0\) but \(\omega(E) = 0\).

- Connectedness does not guarantee Perron solutions and classical solutions of Dirichlet problem agree in dim \(n \geq 3\).

**Theorem: (Dahlberg 1977)** If \(\Omega \subset \mathbb{R}^n\) is a bounded Lipschitz domain, then \(\omega \in A_\infty(\sigma)\).

- Dahlberg actually proved \(\omega \in B_2(\sigma)\) (\(L^2\) Reverse Hölder)
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Examples of NTA Domains

Smooth Domains

Lipschitz Domains

Quasispheres
  (e.g. snowflake)
Theorem: (David and Jerison 1990) Assume $\Omega \subset \mathbb{R}^n$ is a NTA domain. If $\sigma$ is Ahlfors regular, in the sense that,

$$cr^{n-1} \leq \sigma(B(x, r)) \leq Cr^{n-1} \quad \forall x \in \partial \Omega, \ 0 < r < r_0$$

then $\omega \in A_\infty(\sigma)$.

- Ahlfors regular + NTA = higher dimensional chord arc: David and Jerison is analogue of Lavrentiev’s theorem
- Conclusion $\omega \in A_\infty(\sigma)$ is stronger than $\omega \ll \sigma \ll \omega$.
- Proof is based on approximating Ahlfors regular NTA domains with Lipschitz domains and Dahlberg.
- Semmes proved the same result independently (the same year) on “2-sided NTA” domains using stopping time construction.
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**Theorem:** (Badger). Assume $\Omega \subset \mathbb{R}^n$ is NTA. If $\mathcal{H}^{n-1}(\partial \Omega) < \infty$, then $\sigma \ll \omega$.

Moreover, $\omega \ll \sigma$ if and only if

$$B = \left\{ x \in \partial \Omega : \lim_{r \to 0} \frac{\mathcal{H}^{n-1}(\partial \Omega \cap B(x, r))}{r^{n-1}} = \infty \right\}$$

has harmonic measure zero.

Main challenge is constructing Lipschitz approximations without assuming Ahlfors regular.

**Corollary:** (Badger + Kenig-Preiss-Toro) Assume $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \Omega^+$ are NTA. If $\mathcal{H}^{n-1}(\partial \Omega) < \infty$ and $\omega^+ \ll \omega^- \ll \omega^+$, then $\omega \ll \sigma \ll \omega$. 
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Analogue of F. and M. Riesz ($n \geq 3$)

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David–Jerison Approximation Theorem

A closed set $\Sigma \subset \mathbb{R}^n$ has **big pieces of Lipschitz graphs** if

1. Surface measure on $\Sigma$ is Ahlfors regular.

2. For every $x \in \Sigma$ and $0 < r < r_0$ there is a Lipschitz graph $\Gamma$ intersecting $\Sigma$ in a big piece:

$$\mathcal{H}^{n-1}(\Gamma \cap \Sigma \cap B(x, r)) \geq \varphi r^{n-1}$$

**Theorem:** (David and Jerison 1990)
Suppose $\Sigma$ satisfies (1) and $\mathbb{R}^n \setminus \Sigma$ satisfies a “two disk” condition. Then $\Sigma$ has BPLG.
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For corkscrew domains (e.g. for NTA domains), do not need $\sigma$ to be Ahlfors regular to build Lipschitz approximations!

- If $\mathcal{H}^{n-1}(\partial \Omega \cap B(x, r)) < \infty$ at a single location $x \in \partial \Omega$ and scale $r > 0$, then can build good Lipschitz approximation to $\Omega \cap B(x, r)$.

- The precise statement can be made quantitative.
Improved Approximation Theorem

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Building a Lipschitz Approximation to $\Omega \cap B(Q, r)$

**Input:**
- Corkscrew Domain $\Omega$
- Location $x \in \partial \Omega$
- Scale $r > 0$
- Upper Bound $\mathcal{H}^{n-1}(\partial \Omega \cap B(x, r)) < \infty$
- Non-tangential Point $a = a(x, r)$.

**Output:**
- Lipschitz Domain $\Omega_L$
- $a \in \Omega_L \subset \Omega \cap B(x, r))$
- Lower Bound $\mathcal{H}^{n-1}(\partial \Omega_L \cap \partial \Omega) \geq \psi r^{n-1}$
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How do you construct good approximations?

**Ingredients in the Proof**

1. Surface measure $\mathcal{H}^{n-1} \subset \partial \Omega$ on a corkscrew domain $\Omega$ is lower Ahlfors regular!

2. Basic Geometric Lemma: Use cones to identify sets contained in Lipschitz graphs.

3. The Maximal Theorem controls “jumps” in $\partial \Omega$. (This idea is due to David and Jerison.)
The Corkscrew Condition

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then $\Omega$ has the **corkscrew condition** if there exists $M > 1$ and $R > 0$ such that:

- For every $Q \in \partial \Omega$, for every $0 < r < R$ there exists a **non-tangential point** $A \in \Omega \cap B(x, r)$ such that

\[
\text{dist}(A, \partial \Omega) \geq \frac{r}{M}.
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No Spikes in the Boundary

It is useful to see how the corkscrew condition may fail.

The domain on the left fails the interior corkscrew condition. The domain on the right fails the exterior corkscrew condition.

Corkscrew Condition $\Rightarrow$ No Spikes in Boundary
Lemma. On a corkscrew domain, $\mathcal{H}^{n-1}(\partial \Omega \cap B(x, r)) \geq \beta r^{n-1}$ for all $x \in \partial \Omega$ and $0 < r < r_0$ automatically!
Lemma. On a corkscrew domain, $\mathcal{H}^{n-1}(\partial \Omega \cap B(x, r)) \geq \beta r^{n-1}$ for all $x \in \partial \Omega$ and $0 < r < r_0$ automatically!
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**Lemma.** On a corkscrew domain, \( \mathcal{H}^{n-1}(\partial \Omega \cap B(x, r)) \geq \beta r^{n-1} \) for all \( x \in \partial \Omega \) and \( 0 < r < r_0 \) automatically!
Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be projection onto first $(n-1)$-coordinates. Let $f : \mathbb{R}^n \to \mathbb{R}$ be projection onto the last coordinate.

Then $\mathcal{C} = \{ y \in \mathbb{R}^n : f(y) \geq h|\pi(x)| \}$ is a cone with slope $h$.

**Lemma.** Let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then

$$\{ y \in \Sigma : (y + \mathcal{C}) \cap \Sigma = \{ y \} \}$$

sits in the graph of a function $F : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\text{Lip}(F) \leq h$. The graph of $F$ sits “above” $\Sigma$. 
Define the maximal function \( H : \mathbb{R}^{n-1} \to [0, \infty] \) by

\[
H(y) = \sup_{I \ni y} \left\{ \frac{\mu(I)}{\mathcal{H}^{n-1}(I)} \right\}
\]

where the sup is taken over all cubes \( I \subset \mathbb{R}^{n-1} \) such that \( y \in I \).

Maximal Theorem:

\[
\lambda_H(N) = \{ x \in \mathbb{R}^{n-1} : H(x) \geq N \}
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\[
\mathcal{H}^{n-1}(\lambda_H(N)) \leq \frac{Cr^{n-1}}{N}
\]
The surface measure of \( \partial \Omega \cap B(x, r) \) over a set \( E \subset \mathbb{R}^{n-1} \) defines a finite measure \( \mu \) on \( \mathbb{R}^{n-1} \).

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The Maximal Function

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**Maximal Theorem:**

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$$\mathcal{H}^{n-1}(\lambda_H(N)) \leq \frac{C r^{n-1}}{N}$$
Big Jumps

If the surface $\partial \Omega \cap \pi^{-1}(I)$ over a cube $I \subset \mathbb{R}^{n-1}$ has a big vertical span relative to the width of $I$, the maximal function is big on $I$.

Lemma. Let $I \subset \mathbb{R}^{n-1}$ cube of length $t$. Suppose one can find line segments 
- $L$ in $B(x, r) \cap \Omega$
- $L'$ in $B(x, r) \cap \Omega^c$

such that 
- $\pi(L)$ and $\pi(L')$ belong to $I$
- $f(L) \cap f(L')$ length $\gg t$.

Then $H(y) \geq N$ for all $y \in I$.

The maximal theorem limits frequency of big vertical jumps in $\partial \Omega$ over $\mathbb{R}^{n-1}$. 
If the surface $\partial \Omega \cap \pi^{-1}(I)$ over a cube $I \subset \mathbb{R}^{n-1}$ has a big vertical span relative to the width of $I$, the maximal function is big on $I$.

**Lemma.** Let $I \subset \mathbb{R}^{n-1}$ cube of length $t$. Suppose one can find line segments

- $L$ in $B(x, r) \cap \Omega$
- $L'$ in $B(x, r) \cap \Omega^c$

such that

- $\pi(L)$ and $\pi(L')$ belong to $I$
- $f(L) \cap f(L')$ length $\gg t$.

Then $H(y) \geq N$ for all $y \in I$.

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Open Problems

Corkscrew Domains

1. If one does assume $\mathcal{H}^{n-1}(\partial \Omega)$ is upper Ahlfors regular, can one improve the Lipschitz constant in the approximation theorem?

Harmonic Measure

2. Does F. and M. Riesz have a full analogue for NTA domains?

**Conjecture.** Let $\Omega \subset \mathbb{R}^n$ be NTA. Assume $\mathcal{H}^{n-1}(\partial \Omega) < \infty$. Then

$$B = \left\{ Q \in \partial \Omega : \lim_{r \downarrow 0} \frac{\mathcal{H}^{n-1}(\partial \Omega \cap B(Q, r))}{r^{n-1}} = \infty \right\}$$

has harmonic measure zero.
Harmonic Measure w/o Exterior Geometry \((n \geq 3)\)

Result without assumptions on exterior geometry of \(\Omega\):

In 2005, Bennewitz and Lewis showed Ahlfors regular + “good Lipschitz approximations” \(\Rightarrow \omega \in \text{weak } A_\infty(\sigma)\).

Hofmann and Martell; Hofmann, Martell and Uriarte-Tuero prove: (in two recent preprints)

**Theorem: (HM–HMU)** Assume \(\Omega \subset \mathbb{R}^n\) is uniform domain and that \(\Omega\) is regular for Dirichlet problem. If \(\sigma\) is Ahlfors regular, then

\[
\omega \in \text{weak } A_\infty(\sigma) \iff \partial \Omega \text{ is UR.}
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**Open Problem**

Under the same hypotheses as the HM–HMU theorem, prove or disprove the following implications:

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