A characterization of the two weight norm inequality for the Hilbert transform

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**$A_p$ Condition**

**Definition**

A weight is $w \geq 0 \in L^1_{loc}(\mathbb{R}^n)$. Notation: $w(E) = \int_E w = |E|_w$. 

$1 < p < \infty$: $w \in A_p$ means there exists $C > 0$ such that for all $Q$,

$$
\left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{\frac{-1}{p-1}} \right)^{p-1} \leq C 
$$  \hspace{1cm} (0.1)

**Theorem (Coifman, Fefferman, Hunt, Muckenhoupt, Wheeden)**

$1 < p < \infty$, $M$ is the Hardy-Littlewood maximal operator

$M : L^p(w) \rightarrow L^p(w) \iff w \in A_p$ i.e. $\int_{\mathbb{R}^n} |Mf|^p w \leq C_p \int_{\mathbb{R}^n} |f|^p w$

($\iff$) is also true for $T$ Calderón-Zygmund operator

($\implies$) Non-degen $T : L^p(\mu) \rightarrow L^p(\mu) \implies d\mu = w \, dx, w \in A_p$.

But ($\implies$) assumes $\mu(B) > 0$ for any ball $B$  \hspace{1cm} (0.2)
Two weights

\[ \int |Th|^p w \leq C \int |h|^p \nu \]

Standard trick:

\[ h = f \sigma \quad \sigma = \nu^{\frac{-1}{p-1}} \quad \left( \implies \sigma^p \nu = \nu^{\frac{-1}{p-1}} \right) \]

Get:

\[ \int |T (f \sigma)|^p w \leq C \int |f|^p \sigma \]

(In general necessary for testing conditions to be sufficient)

Maximal Function

\[ \mathcal{M} \nu (x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |\nu| \]

Supremum over cubes \( Q \) with sides parallel to coordinate axes containing \( x \)

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Theorem on Maximal Function Inequalities

σ, w positive locally finite Borel measures on \( \mathbb{R}^n \)

\( 1 < p < \infty \)

Muckenhoupt '72 - Two Weight \( A_p \) - (Sawyer’s reformulation)

The maximal operator \( M \) satisfies the weak type two weight norm inequality

\[
\| M(f \sigma) \|_{L^p, \infty(\omega)} \equiv \sup_{\lambda > 0} \lambda \left\{ M(f \sigma) > \lambda \right\}^{\frac{1}{p}} \omega \leq C \| f \|_{L^p(\sigma)}, \quad f \in L^p(\sigma),
\]

if and only if the two weight \( A_p \) condition holds for all cubes \( Q \):

\[
\left( \frac{1}{|Q|} \int_Q d\omega \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q d\sigma \right)^{\frac{1}{p'}} \leq C_2.
\]

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Sawyer ’82 - Two Weight Norm Inequality for $\mathcal{M}$

The maximal operator $\mathcal{M}$ satisfies the two weight norm inequality

$$\|\mathcal{M}(f\sigma)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\sigma)}, \quad f \in L^p(\sigma),$$

(0.7)

if and only if for all cubes $Q$,

$$\int_Q \mathcal{M}(\chi_Q\sigma)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x).$$

(0.8)

- Only need to test the strong type inequality for functions of the form $\chi_Q\sigma$.
- The full $L^p(\omega)$ norm of $\mathcal{M}(\chi_Q\sigma)$ need not be evaluated.
- There is a corresponding weak-type interpretation of the two-weight $A_p$ condition (0.5).
Conjecture I

Conjecture

σ and ω positive Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and $T$ a standard singular integral operator on $\mathbb{R}^n$.

The following two statements are equivalent:

$$\int |T(f\sigma)|^p \omega(dx) \leq C \int |f|^p \sigma(dx), \quad f \in C^0_\infty,$$

$$\left\{ \begin{array}{l}
\left[ \frac{1}{|Q|} \int_Q d\omega \right]^{\frac{1}{p}} \left[ \frac{1}{|Q|} \int_Q d\sigma \right]^{\frac{1}{p'}} \leq C,
\int_Q |T\chi_Q\sigma|^p \omega(dx) \leq C' \int_Q \sigma(dx),
\int_Q |T^*\chi_Q\omega|^{p'} \sigma(dx) \leq C'' \int_Q \omega(dx),
\end{array} \right.$$

cubes $Q$.

In our case:

$$\|H(\sigma f)\|_{L^2(\omega)} \leq C \|f\|_{L^2(\sigma)}. \quad (0.10)$$
• Conjecture is most important when $T$ is Hilbert transform, Beurling Transform, or Riesz Transforms.

• This question occurs in different instances: Sarason Conjecture on composition of Hankel operators, semi-commutator of Toeplitz operators, perturbation theory of some self-adjoint operators.

• Conjecture only verified for positive operators: Poisson integrals, fractional integral operators (Sawyer), and well-localized (or band) operators (i.e. operators which in the Haar basis have a matrix supported in a bounded number of “diagonals”) (Nazarov, Treil and Volberg)

• Cotlar and Sadosky proved the two-weight Helson-Szego Theorem: completely solves the $L^2$ case of the Hilbert transform, but not real-variable.
Let $p = 2$ (most relevant case). For an interval $I$ and measure $\omega$, consider “fattened” (Poisson) $A_2$ condition:

$$\sup_{I} P(I, \omega) \cdot P(I, \sigma) = A_2^2 < \infty,$$  \hspace{1cm} (0.11)

where

$$P(I, \omega) \equiv \int_{\mathbb{R}} \frac{|I|}{(|I| + \text{dist}(x, I))^2} \omega(dx) \left( \geq \frac{\int_{I} d\omega}{|I|} \right)$$  \hspace{1cm} (0.12)

Assume $\exists \mathcal{P} < \infty$ s.t. for all intervals $l_0$, and decompositions $\{I_r : r \geq 1\}$ of $I_0$ into disjoint intervals $I_r \subsetneq I_0$ pivotal condition:

$$\sum_{r=1}^{\infty} \omega(I_r) P(I_r, \chi_{I_0} \sigma)^2 \leq \mathcal{P}^2 \sigma(I_0)$$  \hspace{1cm} (0.13)

And dual pivotal condition (reverse roles of $\sigma$ and $\omega)$:

$$\sum_{r=1}^{\infty} \sigma(I_r) P(I_r, \chi_{I_0} \omega)^2 \leq (\mathcal{P}^*)^2 \omega(I_0)$$  \hspace{1cm} (0.14)
There are two obviously necessary conditions for the two weight Hilbert inequality (0.10) to hold (Sawyer-type testing conditions). Uniformly over intervals \( I \):

\[
\int_I |H(1_I \sigma)|^2 \omega(dx) \leq \mathcal{H}^2 \sigma(I), \quad (0.15)
\]

\[
\int_I |H(1_I \omega)|^2 \sigma(dx) \leq (\mathcal{H}^*)^2 \omega(I). \quad (0.16)
\]
Theorem (Nazarov, Treil, Volberg)

Suppose the pair of weights $\sigma, \omega$ satisfy the pivotal condition and the dual pivotal condition, that is, both $P$ and $P^*$ are finite. Then, the two weight inequality for the Hilbert transform

$$\|H(\sigma f)\|_{L^2(\omega)} \leq C \|f\|_{L^2(\sigma)}.$$  \hspace{1cm} (0.17)

holds if and only if the three quantities $A_2$, $H$ and $H^*$ are all finite.

Nazarov, Treil, and Volberg further stated that the pivotal conditions might be necessary for the Hilbert transform two weight bound (0.17).
Further conditions I

- **Energy condition.** This is key improvement with respect to NTV. Condition that accounts for the distribution / concentration of the weights inside intervals. Given interval $I$, define

$$E(I, \omega) \equiv \left( \mathbb{E}_I \omega(dx) \mathbb{E}_I \omega(dx') \left( \frac{|x - x'|}{|I|} \right)^2 \right)^{1/2}. \quad (0.18)$$

Note $E(I, \omega) \leq 1$, but can be quite small if $\omega$ is highly concentrated inside $I$. E.g. $E(I, \omega) = 0$ if $\omega = \delta_a$ for $a \in I$. The energy functional $E(I, \omega)$ improves “smoothness of the kernel” estimates.

Here,

$$\mathbb{E}_I^\omega \phi \equiv \omega(I)^{-1} \int_I \phi \omega \, dx, \quad (0.19)$$

thus, $\mathbb{E}_I^\omega f$ is the average value of $f$ with respect to the weight $\omega$ on the interval $I$. 

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Two weight norm inequality Hilbert transform
For a parameter $0 \leq \varepsilon \leq 2$, the $\varepsilon$-energy condition is that there exists $E_\varepsilon < \infty$ s.t.

$$\sum_{r=1}^{\infty} \omega(I_r)[E(I_r, \omega)]^\varepsilon [P(I_r, \chi_{I_0} \sigma)]^2 \leq E_\varepsilon^2 \sigma(I_0). \quad (0.20)$$

We require that (0.20) hold for all intervals $I_0$, and strict partitions $\{I_r : r \geq 1\}$ of $I_0$.

Let $E_\varepsilon$ be the smallest constant for which these inequalities are uniformly true. $E_\varepsilon^*$ is the best constant in the same inequality with the roles of $\omega$ and $\sigma$ reversed.
The $\varepsilon$-energy condition (0.20) on a pair of measures increases in strength as $\varepsilon$ decreases. The ($\varepsilon = 2$)-energy condition is necessary: $\mathcal{E}_2 \lesssim A_2 + \mathcal{H}$, and similarly for $\mathcal{E}_2^*$. Proof of this fact uses that $\nu(I) = 0$ implies

$$P(I; \nu) \lesssim |I| \inf_{x, y \in I} \frac{H_{\nu}(x) - H_{\nu}(y)}{x - y}. \quad (0.21)$$

That the constant $\mathcal{E}_0 < \infty$ (i.e. $E(I_r, \omega) \to 1$ in (0.20)) is the pivotal condition (0.13) introduced by Nazarov, Treil and Volberg.
Main Theorem (Lacey, Sawyer, UT ’10)

Let $\omega$ and $\sigma$ be locally finite positive Borel measures on $\mathbb{R}$ with no point masses in common, $\omega(\{x\}) \sigma(\{x\}) = 0$ for all $x \in \mathbb{R}$. Suppose that for some $0 \leq \varepsilon < 2$, $\mathcal{E}_\varepsilon + \mathcal{E}_\varepsilon^* < \infty$. Then

$$\|H(\sigma f)\|_{L^2(\omega)} \leq C \|f\|_{L^2(\sigma)}$$

holds if and only if

$$\sup_{I} P(I, \omega) \cdot P(I, \sigma) = A_2^2 < \infty, \quad (0.11)$$

$$\int_I |H(1_I \sigma)|^2 \omega(dx) \leq \mathcal{H}^2 \sigma(I), \quad (0.15)$$

$$\int_I |H(1_I \omega)|^2 \sigma(dx) \leq (\mathcal{H}^* )^2 \omega(I) \quad (0.16).$$

Moreover,

$$\|H(\cdot \sigma)\|_{L^2(\sigma) \to L^2(\omega)} \lesssim \max \{A_2, \mathcal{H}, \mathcal{H}^*, \mathcal{E}_\varepsilon, \mathcal{E}_\varepsilon^* \}.$$
Recall middle-third Cantor set $E$ and Cantor measure $\omega$.

$\omega$ assigns equal weight to these intervals.
\[ \sigma = \sum_{k,j} s_k \delta_{z_{k,j}}, \quad z_{k,j} \in I_{k,j}. \]

\( s_k \) SATISFY A PRECURSOR TO THE \( A_2 \) CONDITION:

\[ \frac{s_k \omega(3l_{k,j})}{|3l_{k,j}|^2} = 1, \quad s_k = \frac{2^k}{3^{2k}}. \]
Place the $z_{k,j}$ at the center of the $I_{k,j}$:

For this $\sigma$, the pair $(\omega, \sigma)$ satisfies the $A_2$ condition (0.11), the forward testing condition (0.15), but fails the backwards testing condition (0.16). (Alternative example to Nazarov-Volberg and Nikolski-Treil.)

\[
\int_{I} |H\omega 1_I|^{2} \sigma(dx) \leq (\mathcal{H}^*)^2 \omega(I) \quad (0.16)
\]
Consider \( H\omega \) on the intervals \( I_{k,j} \):

Place the \( z_k \), at the unique zeros of \( H\omega \) on \( I_{k,j} \)!

This pair of weights DOES satisfy the two-weight inequality for the Hilbert transform (0.17), but DOES NOT satisfy the dual pivotal condition (0.14).
How does the energy condition appear in the sufficiency proof? For a toy model case, containing idea, recall the familiar argument (with Lebesgue measure): $c_j$ center of $I_j$,

$$\int b_j \, dx = 0 \ , \ \text{supp} \ (b_j) \subseteq I_j \quad \text{(bad function)}$$

Matters reduced to

$$\int_{\mathbb{R} \setminus 2I_j} |Hb_j(y)| \, dy = \int_{\mathbb{R} \setminus 2I_j} \left| \int_{\mathbb{R} \setminus 2I_j} \frac{b_j(x)}{x - y} \, dx \right| \, dy$$

$$= \int_{\mathbb{R} \setminus 2I_j} \left| \int_{I_j} b_j(x) \left( \frac{1}{x - y} - \frac{1}{x' - y} \right) \, dx \right| \, dy \quad (x' \in I_j, \ \text{say } x' = c_j)$$

$$\leq \int_{I_j} |b_j(x)| \int_{\mathbb{R} \setminus 2I_j} \frac{|I_j|}{|y - x'|^2} \, dy \, dx$$

$$\lesssim \int_{I_j} |b_j(x)| \, dx \ P(I_j, \chi_{\mathbb{R} \setminus I_j} \, dy)$$

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Now the weighted version:

$$\int b_j \, d\omega(x) = 0, \supp (\omega b_j) \subseteq I_j \quad \text{bad function}$$

$$\int_{\mathbb{R} \setminus 2I_j} |H(\omega b_j)(y)| \, d\sigma(y) = \int_{\mathbb{R} \setminus 2I_j} \left| \int \frac{b_j(x)}{x-y} \, d\omega(x) \right| \, d\sigma(y)$$

$$= \int_{\mathbb{R} \setminus 2I_j} \left| \int_{I_j} b_j(x) \mathbb{E}^{\omega(dx')}_{l_j} \left( \frac{1}{x-y} - \frac{1}{x'-y} \right) \, d\omega(x) \right| \, d\sigma(y)$$

( if $\omega = \delta_a, a \in I_j$, then $= 0$!! Choice of $x' \in I_j$ now matters!!)

$$\left( \text{write} \left| \frac{1}{x-y} - \frac{1}{x'-y} \right| = \frac{|x-x'|}{|l_j|} \frac{|l_j|}{|(x-y)(x'-y)|} \right)$$

$$\leq \int_{l_j} |b_j(x)| \mathbb{E}^{\omega(dx')}_{l_j} \left( \frac{|x-x'|}{|l_j|} \right) \omega(dx) \, P(l_j, \chi_{\mathbb{R} \setminus l_j} \sigma(dy))$$
Last line of previous slide was:

\[
\int_{I_j} |b_j(x)| \mathbb{E}_{I_j}^{\omega(dx')} \left( \frac{|x - x'|}{|I_j|} \right) \omega(dx) \, P(l_j, \chi_{\mathbb{R}\setminus I_j} \sigma(dy))
\]

\[
\leq \|b_j\|_{L^2(\omega)} \left\{ \int_{I_j} \left[ \mathbb{E}_{I_j}^{\omega(dx')} \left( \frac{|x - x'|}{|I_j|} \right) \right]^2 \omega(dx) \right\}^{1/2} \, P(l_j, \chi_{\mathbb{R}\setminus I_j} \sigma(dy))
\]

\[
= \|b_j\|_{L^2(\omega)} \omega(l_j)^{1/2} \, E(l_j, \omega) \, P(l_j, \chi_{\mathbb{R}\setminus I_j} \sigma(dy))
\]

And now insert this idea into Nazarov-Treil-Volberg's huge machine...
\[ \langle H(\sigma f), \phi \rangle_\omega \]

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Nazarov-Treil-Volberg approach: consider the bilinear form
\[ B(f, \phi) = \langle H_\sigma f, \phi \rangle_w = \langle H(\sigma f), \phi \rangle_w. \]

 Appropriately (randomly) choose a dyadic grid \( D \), define Haar bases adapted to the two weights. E.g. \( \{ h^w_J : J \in D \} \), for the weight \( w \):

Given a dyadic interval \( I \) (\( I \in D \)) and a weight \( w \), if \( w(I^-)w(I^+) > 0 \), (otherwise \( h^w_I = 0 \)),

- \( \text{supp}(h^w_I) = I \),
- \( h^w_I \) is constant on \( I^- \) and \( I^+ \),
- \( \int_I h^w_I \, dw = 0 \),
- \( \int_I (h^w_I)^2 \, dw = 1 \).

\[ h^w_I = \frac{1}{w(I)^{1/2}} \left[ \left( \frac{w(I^+)}{w(I^-)} \right)^{1/2} \chi_{I^-} - \left( \frac{w(I^-)}{w(I^+)} \right)^{1/2} \chi_{I^+} \right] \]
The set $\{h_I^w\}_{I \in \mathcal{D}}$ forms a basis of $L^2(w)$.

Notation

$$\Delta_I^w f = \left( \int_I fh_I^w \, dw \right) \cdot h_I^w = \langle f, h_I^w \rangle_w h_I^w$$

Note that (martingale difference)

$$\Delta_I^w f = \chi_I^+ E_{I^+}^w f + \chi_I^- E_{I^-}^w f - \chi_I E_I^w f$$

Then

$$f = \sum_I \Delta_I^w f$$

and

$$\|f\|_{L^2(w)}^2 = \sum_{I \in \mathcal{D}} |\langle f, h_I^w \rangle_w|^2$$
Then

\[ B(f, \phi) = \sum_{I \in D} \sum_{J \in D} \langle f, h^\sigma_I \rangle_\sigma \langle H \sigma h^\sigma_I, h^w_J \rangle_w \langle \phi, h^w_J \rangle_w. \]

Motivated by $T_1$ theorem, split the sum, according to the whether or not $|I|$ is bigger than $|J|$. This creates two forms, called $B_{abv}(f, \phi)$ and $B_{blw}(f, \phi)$, plus diagonal term.

\[ B_{blw}(f, \phi) \approx \sum_{I \in D} \sum_{J \in D : |J| < |I|} \langle f, h^\sigma_I \rangle_\sigma \langle H \sigma h^\sigma_I, h^w_J \rangle_w \langle \phi, h^w_J \rangle_w. \]

Note that

\[ |B(f, \phi) - \{B_{blw}(f, \phi) + B_{abv}(f, \phi)\}| = |\text{Diagonal term}| \leq \]

\[ \leq (A_2 + \mathcal{H} + \mathcal{H}^*) \| f \|_\sigma \| \phi \|_w \]
Previous proof assumes conditions that guarantee that there is $C < \infty$ such that

$$|B_{\text{blw}}(f, \phi)| \leq C \| f \|_\sigma \| \phi \|_w,$$

which can be verified in any standard $T1$ theorem. But it is not clear that in the two weight setting the boundedness of $B(f, \phi)$ implies that of $B_{\text{blw}}(f, \phi)$. If this implication does not hold for all pairs of weights, then the current approach can not resolve the conjecture!
Write the norm of the bilinear form $B_{blw}$ as $B_{blw}$.

**Main Theorem (Lacey, Sawyer, Shen, UT ’11)**

The following inequalities and their duals hold, for any pair of weights $w, \sigma$ which do not share a common point mass.

\[ B_{blw} + \mathcal{H} + A_2^{1/2} \simeq \mathcal{H} + F + BF + A_2^{1/2}. \]  

That is, modulo the Nazarov-Treil-Volberg conjecture, the above and below splitting is completely understood in terms of the functional energy $F$ and bounded fluctuation $BF$ conditions.

**Theorem (Lacey, Sawyer, Shen, UT ’11)**

Let $(w, \sigma)$ be any pair of weights with $\sigma$ doubling. Then the two weight inequality (0.17) holds if and only if the quantities $A_2$, $\mathcal{H}$, $\mathcal{H}^*$, $F$ and $BF$ are all finite.
DEFINITION

Given interval $I^0$, set $\mathcal{F}(I_0)$ to be the maximal dyadic subintervals $F$ such that $\mathbb{E}_F^\sigma |f| > 4\mathbb{E}_{I_0}^\sigma |f|$.

Set $\mathcal{F}_0 = \{I^0\}$, and inductively set $\mathcal{F}_{j+1} := \bigcup_{F \in \mathcal{F}_j} \mathcal{F}(F)$.

The collection of $f$-stopping intervals is $\mathcal{F} := \bigcup_{j=0}^\infty \mathcal{F}_j$.

Take $f$ non-negative and supported on an interval $I^0$, and $f$-stopping intervals as above. Let $\{g_F : F \in \mathcal{F}\}$ be a collection of functions in $L^2(w)$ which is $\mathcal{F}$-adapted to $F$, i.e., essentially, for each $F$,

1. $g_F$ is supported on $F$ and constant on $F' \in \text{Child}_\mathcal{F}(F)$;
2. $\mathbb{E}_{j^*}^w g_F = 0$ for each $J^* \in \mathcal{J}^*(F)$, where $\mathcal{J}^*(F)$ are the maximal dyadic $J \subset F$ such that $3J \subset F$. 

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Let $F$ be the smallest constant in the inequality below, holding for all non-negative $f \in L^2(\sigma)$, and collections $\{g_F\}$ as just described.

$$\sum_{F \in \mathcal{F}} \sum_{J^* \in J^*(F)} P(f(\mathbb{R} - F)\sigma, J^*) \langle \frac{\chi}{|J^*|}, g_F J^* \rangle_w \leq F \|f\|_\sigma \left[ \sum_{F \in \mathcal{F}} \|g_F\|_w^2 \right]^{1/2}.$$ 

(0.23)

We refer to this as the functional energy condition. Here sets are identified with their characteristic functions.
Write $f \in BF_{\mathcal{F}}(F)$, and say that $f$ is of \textit{bounded fluctuation} if (i) $f$ is supported on $F$, (ii) $f$ is constant on each $F' \in \text{Child}_{\mathcal{F}}(F)$, and (ii) for each dyadic interval $I \subset F$, which is \textit{not contained in} some $F' \in \text{Child}_{\mathcal{F}}(F)$, we have $\mathbb{E}_I |f| \leq 1$.

We then denote as $\mathbf{BF}$ the best constant in the inequality

$$
\left| \sum_{I : \pi_{\mathcal{F}}I = F} \sum_{J : J \subset I, \pi_{\mathcal{F}}J = F} \mathbb{E}_I^{\sigma} \Delta^{\sigma} f \cdot \langle H_\sigma I_J, \Delta^{\sigma} g \rangle_w \right| \leq \mathbf{BF} \{ \sigma(F)^{1/2} + \| f \|_{\sigma} \} \| g \|_w
$$

where $f \in BF_{\mathcal{F}}(F)$, and $g$ is $\mathcal{F}$-adapted to $F$.

Note: the two terms $\sigma(F)^{1/2}$ and $\| f \|_{\sigma}$ on the right above are in general incomparable.

This is the \textit{bounded fluctuation condition}.

The role of the constant one in the inequalities $\mathbb{E}_I^{\sigma} |f| \leq 1$ is immaterial. It can be replaced by any fixed constant.