The Grushin plane admits a BLE into some Euclidean space

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AMS Section meeting, Syracuse University, Oct. 2-3, 2010
Special session on Nonlinear Analysis and Geometry
(Question) Which metric spaces embed bi-Lipschitzly into some finite-dimensional Euclidean space?

Why is this kind of question important?

1. In Analysis: Bi-Lipschitz embeddability into some Euclidean space helps us understand on abstract metric spaces. Bi-Lipschitz mappings are related to problems about differentiability, rectifiability, etc.

2. In Computer science: The size of the bi-Lipschitz constant is related to storage and access issues for large data sets.
1 Motivation and History.
2 Result.
3 Open Question.
Cheeger’s Theorem does not answer about bi-Lipschitz embeddability of $G$ into $\mathbb{R}^N$.

Assouad Embedding Theorem

(Assouad ’83)[1] If $(X, d)$ is doubling metric space, then snowflaked metric space, $(X, d^p)$, admits a bi-Lipschitz embedding in some Euclidean space. Moreover, when there is a bi-Lipschitz embedding from $(\mathbb{R}, d^p)$ into $(\mathbb{R}^n, d_E)$ where $0 < p < 1$, the minimal dimension of Euclidean space is integer strictly greater than $\frac{1}{p}$. 
Pansu Rademacher Theorem

(Pansu ’89)[4] Let $N$ and $N'$ be Carnot groups (simply connected nilpotent Lie group). Every Lipschitz mapping $f$ between open sets in $N$ and $N'$ is differentiable almost everywhere. Moreover, the differential $df_y(x) = \lim_{t \to 0} \delta_t^{-1}(f(y))^{-1}f(y\delta_t(x))$ is a group homomorphism almost everywhere $\Rightarrow$ The Heisenberg group $\mathbb{H}$ does not admit a bi-Lipschitz embedding into any Euclidean space.
Cheeger’s Rademacher Theorem

(Cheeger ’99)[2] The metric space $X = (X, d, \mu)$ which is doubling measure space supporting the $p$-Poincaré inequality for some $p$ has strong measurable differential structure.

1. Each $X_\alpha$ is measurable subset of $X$ with $\mu(X_\alpha) > 0$ and $\mu(X \setminus \bigcup (X_\alpha)) = 0$

2. Each $\pi_\alpha$ is $N(\alpha)$-tuple of Lipschitz functions, for some $N(\alpha) \in \mathbb{N}$, where $N(\alpha)$ is bounded from above independently of $\alpha$.

3. Given a Lipschitz function $f : X \longrightarrow \mathbb{R}$, there exists a function $df^\alpha : X_\alpha \longrightarrow \mathbb{R}^{N(\alpha)}$ so that

$$\limsup_{y \to x} \frac{|f(y) - f(x) - df^\alpha(x) \cdot (x_\alpha(y) - x_\alpha(x))|}{d(x, y)} = 0 \text{ for a.e } x \in X_\alpha.$$ 

Moreover, if $X$ admits a bi-Lipschitz embedding in some finite-dimensional Euclidean space, then at almost every point $z \in X_\alpha$, where $X_\alpha$ is a coordinate patch, the tangent cone $T_z(X_\alpha)$ is bi-Lipschitz to $\mathbb{R}^{N(\alpha)}$. 
Application of Cheeger Theorem to $\mathbb{H}$

The Heisenberg group $\mathbb{H} = (\mathbb{H}, d_{cc}, \mu = \text{Hausdorff measure})$ is a doubling measure space satisfying p-Poincaré inequality.

1. Cheeger’s “coordinate chart” is $(\mathbb{H}, \pi_1, \pi_2)$, where $\pi_1(x, y, t) = x$ and $\pi_2(x, y, t) = y$.

2. Assume $\exists \text{ BLE:} \mathbb{H} \rightarrow \mathbb{R}^N$ for some $N$.
   $\Rightarrow \exists \text{ BL: } \text{Tangent cone } T_z \mathbb{H} = \mathbb{H} \rightarrow \mathbb{R}^2$ for almost every $z \in \mathbb{H}$, which is impossible.
   Thus, $\mathbb{H}$ does not admit a BLE into any Euclidean space.
Can we apply Cheeger’s Theorem to the Grushin plane?
The Grushin plane $\mathbb{G}$

**Definition.** The Grushin plane $\mathbb{G}$ is the plane $\mathbb{R}^2$ with horizontal distribution spanned by $X_1 = \frac{\partial}{\partial x}$ and $X_2 = x \cdot \frac{\partial}{\partial y}$.

1. $[X_1, X_2] = \frac{\partial}{\partial y} \implies$ every two points can be joined by horizontal curve.
2. $p = (x, y), x \neq 0$ is regular point and $x = 0$ is the singular line.
3. Metric tangent cone to $\mathbb{G}$ at singular point $= \mathbb{G}$.
   Metric tangent cone to $\mathbb{G}$ at regular point $= \mathbb{R}^2$
4. $\dim_H(\mathbb{G}) = 2$
The Grushin plane $\mathbb{G}$

**Distance estimate**

1. dilation $\delta_\lambda(x, y) = (\lambda x, \lambda^2 y) \Rightarrow$ On the singular line $x = 0$, $d_\mathbb{G}((0, y_1), (0, y_2)) = \sqrt{|y_1 - y_2|} d_\mathbb{G}((0, 0), (0, 1))$

2. Outside of the singular line $x = 0$, the sub-Riemannian metric is Riemannian. For any points $p = (x_1, y_1)$ and $q = (x_2, y_2)$,

\[
\frac{1}{2} \left( |x_1 - x_2| + \frac{|y_1 - y_2|}{\sqrt{\min(|x_1|, |x_2|)^2 + 4|y_1 - y_2|}} \right) \leq d_\mathbb{G}(p, q)
\]

(1)

\[
4(|x_1 - x_2| + \sqrt{|y_1 - y_2|}) \geq d_\mathbb{G}(p, q)
\]

(2)
Cheeger’s Theorem does not answer about bi-Lipschitz embeddability of \( G \) into \( \mathbb{R}^N \)

The Grushin plane \( G = (G, d_G, \mu = \text{Lebesgue measure}) \) locally is a doubling measure space satisfying \( p \)-Poincaré inequality[3].

1. Cheeger’s “coordinate charts” \( (K, \pi_1, \pi_2) \), where \( \pi_1(x, y) = x \) and \( \pi_2(x, y) = y \) for any compact \( K \subset G \).

2. Since every Tangent cone \( T_pK \) is bi-Lipschitz to \( \mathbb{R}^2 \) for every \( p \in K \setminus A \) and \( \mu(A) = 0 \), Cheeger’s theorem does not answer about whether or not the Grushin plane locally embeds bi-Lipschitzly into some \( \mathbb{R}^N \).
The Grushin plane equipped with Carnot-Carathéodory distance admits a bi-Lipschitz embedding into some Euclidean space. The bi-Lipschitz constant and dimension of the Euclidean space depends only on a bi-Lipschitz constant of a bi-Lipschitz embedding of the singular line into $\mathbb{R}^3$. 
Outline of Proof

1. Bi-Lipschitz embedding of the singular line.
2. Extend to global Lipschitz map.
3. Make co-Lipschitz using local and large scale argument in the Whitney distance manner.
From Assouad's embedding theorem and distance estimate on the singular line, we have that \((A = y\text{-axis, } d_G)\) admits a BLE \(f : (A, d_G) \longrightarrow \mathbb{R}^3\). i.e. there exists a constant \(L > 1\) such that

\[
\frac{1}{L} \cdot d_G(p, q) \leq |f(p) - f(q)| \leq L \cdot d_G(p, q), \quad \text{where } p, q \in A.
\]
Bi-Lipschitz embedding on the singular line

From Assouad’s embedding theorem and distance estimate on the singular line, we have that \((A = y\text{-axis}, d_G)\) admits a BLE \(f : (A, d_G) \rightarrow \mathbb{R}^3\). i.e. there exists a constant \(L > 1\) such that \(\frac{1}{L} \cdot d_G(p, q) \leq |f(p) - f(q)| \leq L \cdot d_G(p, q)\), where \(p, q \in A\).
Whitney decomposition $W_\Omega$ of $\Omega = \mathbb{G} \setminus A$

We now want to extend $f : A \rightarrow \mathbb{R}^3$ to a Lipschitz map $g : \mathbb{G} \rightarrow \mathbb{R}^3$.

Let $A$ be the y-axis in $G$. Then its complement $\Omega$ is the union of boxes $Q_{n,k} = \left[ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \times \left[ \frac{k}{2^{2n}}, \frac{k}{2^{2n}} \right]$ and $Q'_{n,k} = \left[ -\frac{1}{2^{n-1}}, -\frac{1}{2^n} \right] \times \left[ \frac{k}{2^{2n}}, \frac{k}{2^{2n}} \right]$ for any integers $n$ and $k$, whose interiors are mutually disjoint and whose diameters are approximately proportional to their distances from $A$. More precisely,

1. $\Omega = \bigcup_{n \in \mathbb{Z}} (W_\Omega(n) \cup W_\Omega(\bar{n}))$ where $W_\Omega(n) = \{ Q_{n,k} \}_{k \in \mathbb{N}}$ and $W_\Omega(\bar{n}) = \{ Q'_{n,k} \}_{k \in \mathbb{N}}$.
2. The interiors are mutually disjoint.
3. $\text{dist}_G(Q, A) \leq \text{diam}_G(Q) \leq 8 \text{dist}_G(Q, A)$. 
A partition of unity

\[ Q^* = \text{the set of Whitney boxes which touch } Q \]
\[ Q^{**} = \text{the set of Whitney boxes which touch } Q^*. \]

We can associate a partition of unity \( \{ \varphi_{Q^*} \} \) with the following properties.

1. \( 0 \leq \varphi_{Q^*} \leq 1 \)
2. \( \varphi_{Q^*}|_{Q^*} \geq \frac{1}{C_1} > 0 \) and \( \varphi_{Q^*}|_{(Q^{**})^c} = 0 \).
3. \( \varphi_{Q^*} \) is Lipschitz with constant \( C_2/\text{diam}_G(Q) \)
4. For every \( p \in \Omega \), we have \( \varphi_{Q^*}(p) \neq 0 \) for at most \( C_3 \) boxes \( Q \)
5. \( \sum_{Q \in \mathcal{W}_\Omega} \varphi_{Q^*} = 1. \)
Whitney Lipschitz extension of $f$ on $\Omega$

We can now define Lipschitz extension map $g$ of $f$. Let $A$ be the singular line of the Grushin plane $G$ and $f$ be the bi-Lipschitz embedding of $(A, d_G)$ into $\mathbb{R}^3$.

$$g(p) = \begin{cases} 
\sum_{Q \in W_\Omega} f(z_Q) \varphi_Q^*(p), & p \in \Omega, z_Q \in A \text{ such that } \text{dist}_G(A, Q) = \text{dist}_G(z_Q, Q);

f(p) & p \in A.
\end{cases}$$

(3)

is a Lipschitz extension of $f$ on the Grushin plane.

**Remark** $g$ is not globally co-Lipschitz.
**Whitney distance map**

**Definition.** The Whitney distance map $d_W$ on $W_\Omega \times W_\Omega$ is defined by

$$d_W(Q, R) = \frac{\text{dist}_G(Q, R)}{\min(\text{diam}_G(Q), \text{diam}_G(R))}.$$  

By using this definition, we break $\Omega$ into two parts and construct co-Lipschitz maps on those parts. More precisely, we make co-Lipschitz maps on

1. for any $p \in Q$ and $q \in R$ such that $d_W(Q, R) > 8L^2$. (W-large scale co-Lipschitz)
2. for any $p \in Q$ and $q \in R$ such that $d_W(Q, R) \leq 8L^2$. (W-local co-Lipschitz)
"W-Large" scale co-Lipschitz and globally Lipschitz map

Lemma

For any $p \in Q$ and $q \in R$ where $d_W(Q, R) > 8L^2$, the Lipschitz extension map $g$ and $\text{dist}_G(\cdot, A)$ guarantee $W$-large scale co-Lipschitz bounds. More precisely,

1. If \[
\frac{\text{dist}_G(Q, R)}{\max(\text{diam}_G(Q), \text{diam}_G(R))} \geq 4L^2,
\]
   then $|g(p) - g(q)| \geq C(L) \cdot d_G(p, q)$.

2. If \[
\frac{\text{dist}_G(Q, R)}{\max(\text{diam}_G(Q), \text{diam}_G(R))} \leq 4L^2,
\]
   then $|\text{dist}_G(p, A) - \text{dist}_G(q, A)| \geq C(L) \cdot d_G(p, q)$.

Thus, $g \times \text{dist}(\cdot, A)$ is the desired $W$-large scale co-Lipschitz and global Lipschitz map on $G$. 
Coloring map on Whitney decomposition

We remain construction of co-Lipschitz map on \( d_W(Q, R) < 8L^2 \). The dimension 4 is not enough to construct co-Lipschitz map on this part. Thus, we use a coloring map that gives additional dimension of the Euclidean space.

**Lemma**

For any \( Q \in W_\Omega \), \( \#\{ R \in W_\Omega \mid d_W(Q, R) < 8L^2 \} \) is finite, \( m(L) \).

**Lemma**

There exists a coloring map

\[
K : W_\Omega \longrightarrow \{1, 2, 3, \ldots, M\} \text{ where } M \geq m(m-1)
\]

such that each box in \( W \)-ball of radius \( 8L^2 \) has different color. (i.e if \( R', R'' \) have \( d_W(R', R'') < 8L^2 \), then \( K(R') \neq K(R'') \)).
Since the complement of the singular line is Riemannian, we can construct local patches of bi-Lipschitz embeddings into $\mathbb{R}^2$ with uniform bi-Lipschitz constant (independent of Whitney boxes). Next we will put together all patches to make “$W$-Locally co-Lipschitz” map by assigning different colors to each Whitney boxes.
“W-Locally” co-Lipschitz and globally Lipschitz map

Lemma

When we define the map $h$ from $\Omega$ into $\mathbb{R}^2$ as the following:

$$h(x, y) = (x, \text{diam}_G(Q)^{-1} \cdot y)$$

then, the restriction map on each $Q^*$, $h|_{Q^*}$, is a bi-Lipschitz from $Q^*$ into $\mathbb{R}^2$ with uniform bi-Lipschitz constant. Also, we have a map $\hat{h}$ from $\Omega$ to $\mathbb{R}^2$ so that there exists $M_1 > 1$ which is independent of Whitney boxes such that the restriction on each $Q^*$, $\hat{h}|_{Q^*}(Q^*) \subset B(0, M_1 \text{diam}_G(Q)) \setminus B(0, \frac{1}{M_1} \text{diam}_G(Q))$. 
"W-Locally" co-Lipschitz and globally Lipschitz map

**Lemma**

The following map $H$ from $\Omega$ into $(\mathbb{R}^2)^M$ given by

$$H(p) = \sum_{Q \in \mathcal{W}_\Omega} \tilde{h}_Q^*(p) \otimes e_K(Q),$$

is a global Lipschitz and $W$-local co-Lipschitz map where $\tilde{h}_Q^*$ is defined by $\hat{h} \cdot \varphi_Q^*$. The ($W$-local) bi-Lipschitz constant depends only on $L$. That is,

$$|H(p) - H(q)| \geq C(L) \cdot d_G(p, q)$$

for any $p \in Q$, $q \in R$ where $d_W(Q, R) < 8L^2$. 
Bi-Lipschitz embedding on the Grushin plane

Theorem

\[ F : \mathbb{G} \rightarrow \mathbb{R}^3 \times \mathbb{R}^{2M} \times \mathbb{R} \] defined by

\[ F(p) = g(p) \times H(p) \times \text{dist}(p, A) \quad \text{for } p \in \mathbb{G}. \] (5)

is a bi-Lipschitz embedding. (Bi-Lipschitz constant depends on \(L\))

1. \(g, H, \) and \(\text{dist}(\cdot, A)\) are Lipschitz maps \(\Rightarrow F\) is Lipschitz.
2. “\(W\)-locally” \[ |F(p) - F(q)| \geq |H(p) - H(q)| \geq C(L) \cdot d_{\mathbb{G}}(p, q). \]
3. “\(W\)-large scale”

\[ |F(p) - F(q)| \geq |g(p, A) - g(q, A)| \geq C(L) \cdot d_{\mathbb{G}}(p, q) \]

\[ |F(p) - F(q)| \geq |\text{dist}(p, A) - \text{dist}(q, A)| \geq C(L) \cdot d_{\mathbb{G}}(p, q). \]

\(\Rightarrow F\) is co-Lipschitz.
The dimension $2M + 4$ of the Euclidean space depends only on the bi-Lipschitz constant $L$ of the bi-Lipschitz embedding $f$ of the singular line. However, the number of colors $M(L) \sim L^8(\log_2 L)^4$ is somewhat large. The question, what is the minimal dimension of Euclidean space into which the Grushin plane bi-Lipschitzly embeds, remains open.
References

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Thank You!
Conjecture

$(X, d)$ has S-property:

1. $Y \subset X$ and $\exists L_1$-BL embedding from $Y$ into $R^{M_1}$.
2. $\Omega = X \setminus Y$ has Whitney decomposition
3. $\Omega = \bigcup Q$ and $\exists L_2$-BL : $Q^* \hookrightarrow R^{M_2}$, $L_2, M_2$ in uniform manner.
4. The number of Whitney boxes within $W-8L_1^2$ ball is finite.
   
   \{ $X$ is doubling metric space(?) \}

$\Rightarrow$ There exists a $L_3$- BL embedding from $(X, d)$ in $R^{M_3}$. where $L_3$ and $M_3$ depends on $L_1, L_2, M_1, M_2$. \{doubling constant(?)\}